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A Method for Analyzing Survivability in the Context of a One-on-One Engagement

by Jeffrey A. Smith

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14. ABSTRACT

We imagine a one-on-one engagement between two adversarial combat platforms, and we want to know who is victorious and who is not. The simple answer is that the victor lives while the vanquished dies. However, if our desire is meaningful analysis platform survivability, then this answer is insufficient. Our answer should reflect a more practical view of victory and defeat; one that illuminates the fate of each platform if we could repeatedly conduct this same engagement. Given that the current paradigm for platform survivability is layered survivability, we ask if there is a unified framework available with which to analyze the full range of survivability options layered survivability suggests. This paper proposes a framework built upon a mathematical construct called the stochastic duel. To construct our duel, we first consider one platform engaging a passive target and model that process via the techniques of renewal theory. From there, we model the one-on-one engagement as a series system, which allows us to apply the method of competing risks. With competing risk framework, we then derive a metric for the probability of survival. Finally, we extend this formalism to include survivability by relying on the concept of layered survivability.

15. SUBJECT TERMS

Survivability, layered, onion, platform, ground, combat, renewal, theory, probability, statistics, survivability measures, competing risk, independence, dependence, lifetime, system, series system

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Summary

Consider a one-on-one engagement between two adversarial combat platforms, and an important outcome of that engagement: who is victorious and who is not. A simple answer is that the victor lives while the vanquished dies, but for a meaningful analysis of platform survivability, this answer is insufficient. A better answer ought to reflect a more practical view of victory and defeat; one that illuminates the fate of each platform if we could repeatedly conduct the same engagement. Given that the current paradigm for platform survivability is one of layered survivability, there is a need for a unified framework within which one can analyze the full range of survivability options layered survivability suggests. This paper sketches such a framework, one built upon a mathematical construct called the stochastic duel. As we construct this duel, we first consider a platform that engages a passive target, and we model this process via the techniques of renewal theory. From there, we model the one-on-one engagement by equating it to a series system, a choice that allows us to apply the method of competing risks. Within the competing risk framework, we then derive a metric for a platform's probability of survival. Finally, we extend this formalism to include survivability measures by relying on the concept of layered survivability.

1. Introduction

This paper considers a simple one-on-one engagement between two combat platforms in an effort to study the outcome: who is victorious and who is not. We find the trivial answer, that the victor lives while the vanquished dies, insufficient for our analytical needs. The answer we seek ought to reflect a more practical view of victory and defeat; one that illuminates the fate of each platform if we repeatedly conducted the same engagement. We shall assume that the victor moves on to other engagements and the vanquished does not; however, we will not assume the destruction of the vanquished or that the victor proceeds unscathed. In other words, we assume that the victor remains mission capable while the vanquished does not. The U.S. Department of Defense Dictionary (2001) defines *mission capable* as the "material condition of an aircraft indicating that it can perform at least one and potentially all of its designated missions." While this definition applies to aircraft, we take this definition to apply equally to ground combat platforms. Accordingly, how can one determine each platform's fate if we could conduct this single engagement repeatedly? We motivate this research with the desire to address this very question; thus, we turn to a mathematical model called the *stochastic duel*.

Stochastic duels, as an analytical approach, were first proposed by Williams and Ancker (1963). Their approach involved two duelists who fired at random intervals. Each duelist possessed unlimited ammunition, fired with a fixed, single shot probability of hit, and the first one to hit the other became the victor. Williams and Ancker used the term *marksman problem* to describe a combatant firing at a passive target until he observes a hit. Then, by assuming stochastic independence between these marksmen, Williams and Ancker turned these marksmen against one another to form what they termed the *fundamental duel*.

We set two goals for this paper: (1) explore the fundamental duel and the conditions that impose stochastic independence, and (2) incorporate survivability measures into the duel. At the conclusion of this paper, we will have built the foundation for a follow on paper that describes our model of the fundamental duel. We structure this paper first to express stochastic duels as renewal processes, next to study the fundamental duel, and finally to add survivability measures to the renewal duel formulation. Consequently, section 2 provides a brief review of renewal theory, and in section 3, we employ renewal theory to model the component processes of stochastic duels as renewal processes. Section 4 constructs the fundamental stochastic duel of Williams and Ancker by framing the duel as a competing risk. In section 5, we employ the mathematics developed in the previous sections to derive solutions for the fundamental duel; solutions that we then validate by comparing them to results obtained by Williams and Ancker. Next, section 6 introduces a concept called layered survivability in order to establish a basis for section 7, which extends our renewal duel formulation to include survivability measures.

2. A Brief View of Renewal Theory

We begin our review by considering a sequence of arbitrary events that occur at random intervals. We assume that one can model these intervals as independent random variables with identical distributions. When the length of the interval between successive events is non-negative with probability one (i.e., almost surely), we term the sequence a *renewal process* (Kao, 1997). As we describe renewal processes, we rely on the standard example of that process: a sequence of light bulbs where each bulb runs consecutively. At time $\boldsymbol{S}_0 = 0,$ we apply power to the first bulb. After a period of operation, the light bulb fails and we let $\boldsymbol{S}_1$ record the epoch of first failure. We immediately replace the bulb with an equivalent, and replicate this process when the bulb fails at $\boldsymbol{S}_2,\ \boldsymbol{S}_3,$ etc. Our desire is two-fold; first, we seek a count of replacements up to some time $\boldsymbol{t}$, and second, we wish to understand how the process behaves as $\boldsymbol{t}$ grows without bound. We set $\boldsymbol{S}_0 = 0$ and let $\{\boldsymbol{S}_i\}_{i=1}^{n}$ denote the times, or equivalently epochs, at which the events occur. We define the interval between successive events to be $\boldsymbol{X}_i = \boldsymbol{S}_i - \boldsymbol{S}_{i-1},$ and assume that the lengths of these intervals are independent and identically distributed (i.i.d.) random variables with $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{X}_i \le \boldsymbol{x}).$ Thus

$$\boldsymbol{S}_n = \sum_{i=1}^{n} \boldsymbol{X}_i \tag{1}$$

is the elapsed time until the occurrence of the $\boldsymbol{n}^{th}$ event. Set

$$\boldsymbol{N}(\boldsymbol{t}) = \max\{\boldsymbol{n} : \boldsymbol{S}_n \le \boldsymbol{t}\} \tag{2}$$

so that $\boldsymbol{N}(\boldsymbol{t})$ counts the events in the interval (0, $\boldsymbol{t}$]. Thus, we refer to $\{\boldsymbol{N}(\boldsymbol{t}), \boldsymbol{t} \ge 0\}$ as the *counting process*, and $\{\boldsymbol{S}_n, \boldsymbol{n} \ge 0\}$ as the *partial sum process*; however, in keeping with common practice, either process may be referred to as the *renewal process* (Karlin and Taylor, 1975; Kao, 1997). We may determine the distribution of $\boldsymbol{N}(\boldsymbol{t})$ from our knowledge of $\boldsymbol{F}(\boldsymbol{x})$ by employing the following relationship given by Karlin and Taylor

$$\boldsymbol{N}(\boldsymbol{t}) \ge \boldsymbol{n} \quad \text{iff} \quad \boldsymbol{S}_n \le \boldsymbol{t}, \tag{3}$$

where *iff* means *if and only if*. An immediate consequence of this relationship is that

$$\begin{aligned} \boldsymbol{P}\left(\boldsymbol{N}(\boldsymbol{t}) \ge \boldsymbol{n}\right) &= \boldsymbol{P}\left(\boldsymbol{S}_n \le \boldsymbol{t}\right) \\ &= \boldsymbol{F}_n(\boldsymbol{t}), \quad \boldsymbol{t} \ge 0, \boldsymbol{n} = 1, 2, \cdots, \end{aligned} \tag{4}$$

where

$$F_n(t) = \begin{cases} \int_0^t F_{n-1}(t-\tau)dF(\tau), & n > 1, \\ F(t), & n = 1. \end{cases} \tag{5}$$

Consequently, Kao (1997) determines that

$$P\left(N(t) = n\right) = \begin{cases} P\left(N(t) \geq n\right) - P\left(N(t) \geq n+1\right), \\ P\left(S_n \leq t\right) - P\left(S_{n+1} \leq t\right), \\ F_n(t) - F_{n+1}(t), & t \geq 0, n = 1, 2, \cdots. \end{cases} \tag{6}$$

We now turn to the renewal function $\boldsymbol{M(t)}$, which we define as

$$M(t) = E\left[N(t)\right]. \tag{7}$$

By standard arguments, various authors rewrite eq 7 as

$$\begin{aligned} M(t) &= \sum_{n=1}^{\infty} nP\left(N(t) = n\right) \\ &= \sum_{n=1}^{\infty} P\left(N(t) \geq n\right) \\ &= \sum_{n=1}^{\infty} F_n(t). \end{aligned} \tag{8}$$

Karlin and Taylor (1975) formally prove convergence of the series

$$M(t) = \sum_{n=1}^{\infty} F_n(t). \tag{9}$$

We can examine the renewal function in more detail by recognizing that the underlying counting process is a probabilistic replica of itself between each event. If we denote the first epoch as X_1, and condition on its occurrence at $X_1 = x$, we can write

$$E\left[N(t) \mid X_1 = x\right] = \begin{cases} 1 + M(t-x), & x \leq t, \\ 0, & x > t, \end{cases} \tag{10}$$

where the quantity $1 + M(t-x)$ counts the first event plus any remaining events in the interval $(\boldsymbol{x}, \boldsymbol{t}]$. Thus, by applying the law of total probability (Bartoszyñski and Niewiadomska-Bugaj, 1996) over the range of arrival times, we find a functional equivalent for the renewal function

$$
\begin{aligned}
\boldsymbol{M}(\boldsymbol{t}) &= \boldsymbol{E}[\boldsymbol{N}(\boldsymbol{t})] \\
&= \int_0^\infty \boldsymbol{E}\left[\boldsymbol{N}(\boldsymbol{t}) \mid \boldsymbol{X}_1 = \boldsymbol{x}\right] \boldsymbol{dF}(\boldsymbol{x}) \\
&= \int_0^t \left[1 + \boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})\right] \boldsymbol{dF}(\boldsymbol{x}) \\
&= \boldsymbol{F}(\boldsymbol{t}) + \int_0^t \boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})\, \boldsymbol{dF}(\boldsymbol{x}).
\end{aligned} \tag{11}
$$

We will find a solution to eq 11 after considering Theorem 1.

Theorem 1
Suppose $\boldsymbol{a}$ is a bounded function. There exists one and only one function $\boldsymbol{A}$ bounded on finite intervals that satisfies

$$\boldsymbol{A}(\boldsymbol{t}) = \boldsymbol{a}(\boldsymbol{t}) + \int_0^t \boldsymbol{A}(\boldsymbol{t}-\tau)\, \boldsymbol{dF}(\tau). \tag{12}$$

This function is

$$\boldsymbol{A}(\boldsymbol{t}) = \boldsymbol{a}(\boldsymbol{t}) + \int_0^t \boldsymbol{a}(\boldsymbol{t}-\tau)\, \boldsymbol{dM}(\tau). \tag{13}$$

where $\boldsymbol{M}(\boldsymbol{t}) = \sum_{\boldsymbol{n}=1}^{\infty} \boldsymbol{F}_{\boldsymbol{n}}(\boldsymbol{t})$ is the renewal function. (Karlin and Taylor, 1975, p. 184-185).

Observe that when $\boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{F}(\boldsymbol{t})$ and $\boldsymbol{A}(\boldsymbol{t}) = \boldsymbol{M}(\boldsymbol{t})$, eq 11 satisfies the requirements of Theorem 1. Thus, we can write the following unique (and equivalent by the property of convolutions) solutions to eq 11

$$
\begin{aligned}
\boldsymbol{M}(\boldsymbol{t}) &= \boldsymbol{F}(\boldsymbol{t}) + \int_0^t \boldsymbol{F}(\boldsymbol{t}-\boldsymbol{x})\, \boldsymbol{dM}(\boldsymbol{x}) \\
&= \boldsymbol{F}(\boldsymbol{t}) + \int_0^t \boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})\, \boldsymbol{dF}(\boldsymbol{x}).
\end{aligned} \tag{14}
$$

Equation 14 completes our review of renewal theory, and we finish the section by noting that in general, closed-form solutions to this equation are only possible with the simplest of inter-arrival distributions. We now begin our study of stochastic duels by employing renewal theory to mathematically express the component processes of the fundamental duel.

3. Stochastic Duels: The Components as Renewal Processes

In this section, we study a one-on-one engagement between two adversaries; one arbitrarily termed the *combatant* and the other the *target* simply to signify that the target is the object of the combatant's attention. The combatant defeats the target by undertaking and successfully completing three consecutive, but distinctly different, processes. To describe these processes, we will introduce some terms as convenient referents for the mathematics. In the first of these processes, the combatant seeks out a target; we incorporate this process into the *generalized marksman problem*. Here, we take up the term "marksman problem" in the sense of Williams and Ancker (1963), but adapt it to reflect our extension to their work. Secondly, once the combatant detects a target, he then engages by loading, aiming, and firing his weapon at the target; we call this process the *marksman problem* to reflect that this process represents the problem as originally conceived by Williams and Ancker. Finally, once the combatant hits the target we want to ascertain the extent to which the target sustains damage. We employ the term *damage process* to describe the accretion of damage by a target with each hit.

In this section, we will first express the marksman problem as a *renewal* process. Then, we will generalize this model to include the process of detection by considering the marksman problem as a *delayed renewal* process. Finally, we will model the damage process via renewal theory. Thus, after finishing this section, we will obtain a complete mathematical model for one-half of a one-on-one duel by joining a generalized marksman problem and a damage process together.

3.1 The Marksman Problem

We begin by considering a combatant as he fires at the strictly passive target. The combatant begins a trial at time zero with weapons unloaded. After a random interval, the combatant loads and fires at the target. Once the combatant fires, he instantaneously observes the outcome of his shot. If he misses the target, he repeats the firing process; however, if he hits the target, he begins a new trial. Figure 1 provides a sample realization of this process where we employ dashed arrows to signify shots that miss, and solid arrows to signify shots that hit. Figure 2 represents the associated counting process for the hits within this sample process. If our sole interest lay in the distribution of the intervals between consecutive *shots* taken in $(0,\boldsymbol{t}]$, we could avail ourselves of the approach outlined in section 2 to determine that distribution. However, in our case, the outcome of each shot is *also* a random process. Our interest, therefore, ought to be in deducing the distribution of intervals between consecutive occurrences of the *same outcome*. Before we can find that distribution, we first make some observations regarding this process and then define some necessary variables.

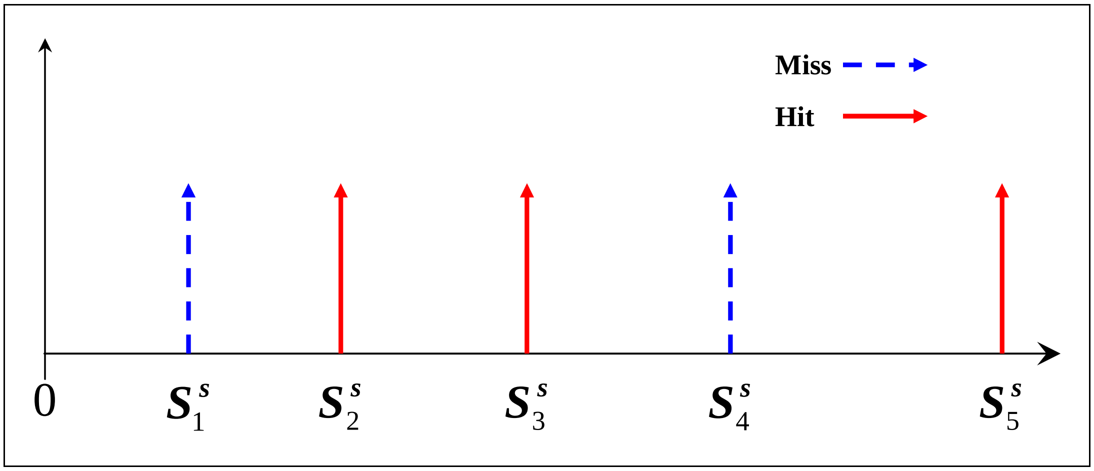

Figure 1. Sample shot history.

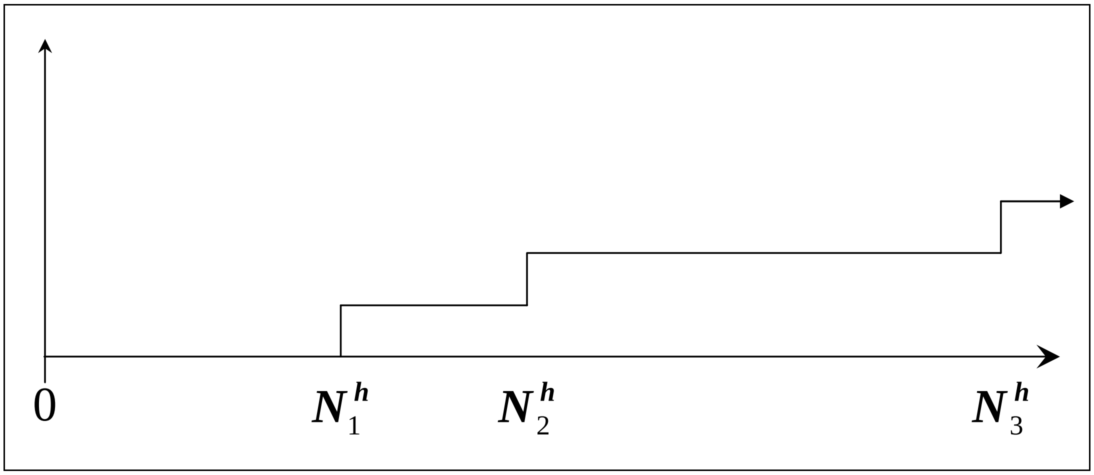

Figure 2. Count of hits as a function of time.

One could view the combatant's sequence of shots and shot outcomes as a stochastic process of the form $\{\boldsymbol{S}_i^s, \boldsymbol{Y}_i\}$, where $\boldsymbol{S}^s$ denotes the firing time of the shot, $\boldsymbol{Y}$ denotes the location of the shot impact (not necessarily on the target), and $\boldsymbol{i}$ indexes the shot. While perhaps desirable, modeling such a process is practically impossible for all but the most simplistic functions. However, if we model this process as $\{\boldsymbol{S}_i^s, \boldsymbol{I}(\boldsymbol{Y}_i)\}$, we can resort to established techniques to find our solutions. Here we employ the indicator function $\boldsymbol{I}(\boldsymbol{Y}_i)$ in the following sense

$$\boldsymbol{I}(\boldsymbol{Y}_i) = \begin{cases} 1, & \text{if the point of shot impact is on the target,} \\ 0, & \text{if the point of shot impact is off the target.} \end{cases}$$

By employing the indicator notation, we recognize that we are dealing with a stochastic process that consists of: (1) a shot process with an embedded (2) hit process. In this process, shot events occur at epochs $\{\boldsymbol{S}_i^s\}_{i=1}^{\infty}$ and hit events occur at epochs $\{\boldsymbol{S}_k^h\}_{k=1}^{\infty}$. We note that for every $\boldsymbol{S}_k^h$, there is one, and only one $\boldsymbol{S}_i^s$, such that $\boldsymbol{S}_k^h = \boldsymbol{S}_i^s$; however, in general the converse is not true.

We start our definitions by setting $S_0^s = S_0^h = 0$ so that we can denote the interval between consecutive shots as $X_i^s = S_i^s - S_{i-1}^s$, and define $X_k^h = S_k^h - S_{k-1}^h$ to be the interval between consecutive hits. Figure 3 depicts the relationship between these variables as they apply to the sample process of figure 1. Consequently, if each $X_i^s \geq 0$ almost surely, we can model this process as a renewal process. We will assume that the intervals X_i^s and outcomes Y_i form an i.i.d. sequence of random variables of finite variance, with distributions $F_X^s(x) = P(X_i^s \leq x)$ and $F_Y(y) = P(Y_i \leq y)$ respectively. We note that while Karlin and Taylor (1975) allow for the possibility that Y_i may depend on X_i^s, we shall assume independence between Y_i and X_i^s, and defer our exploration of dependence.

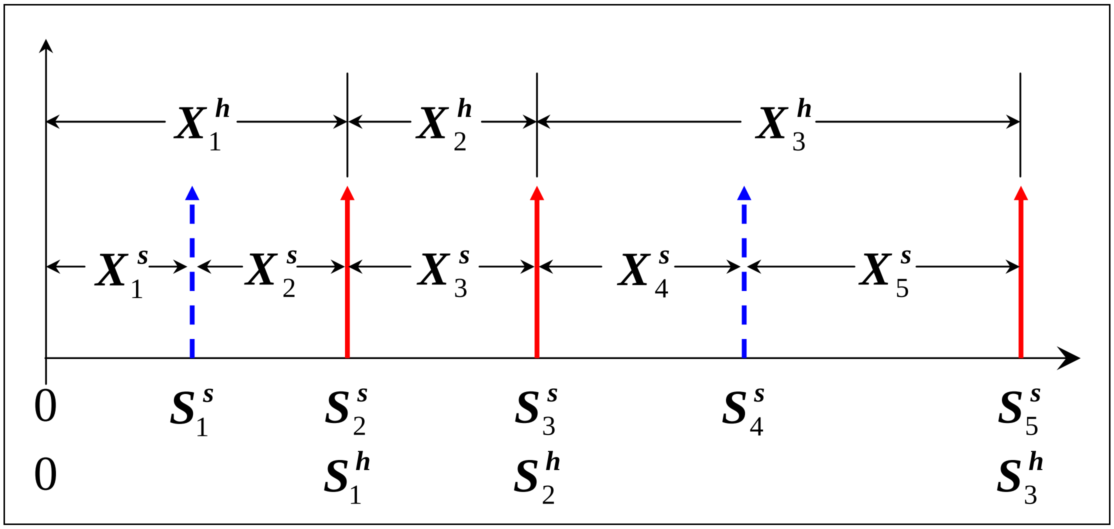

Figure 3. Relationships among key variables.

By employing the indicator function as we did, we restricted the possible outcomes for a given shot to two: either the shot hit or it missed. Thus, we can employ Doob (1994) to define the probability of hit as

$$p_h = \int_{I(Y_i)=1} dF_Y(y). \tag{15}$$

While literature, such as the work of Helgert (1971), Jaiswal (1997) or Przemieniecki (2000), exists on computing the probability of hit for many types of targets, for the purpose of this paper, we shall assume that the probability of hit is both known and constant.

With our definitions complete, we can now develop the marksman problem as a renewal process. Our goal is to find the probability distribution that describes the interval between consecutive hits, a goal we represent by constructing the renewal argument $M(t) = P(X_k^h \leq t)$. Our task is to find a functional equivalent for this argument based upon our knowledge of $F_X^s(x)$. We begin by conditioning on the epoch of the first shot

$$\left[\boldsymbol{M(t)} \mid \boldsymbol{X}_1^s = \boldsymbol{x}\right] = \begin{cases} \{\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{Y}_1), \boldsymbol{X}_1^s = \boldsymbol{x}\}, & \boldsymbol{x} \le \boldsymbol{t}, \\ 0, & \boldsymbol{x} > \boldsymbol{t}. \end{cases} \tag{16}$$

In eq 16, the term $\{\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{Y}_1), \boldsymbol{X}_1^s = \boldsymbol{x}\}$ reflects that our argument necessarily depends on the outcome of the first shot. As noted before, we restricted the range of possible outcomes for a given shot to two: either a hit, which implies the event $\boldsymbol{I}(\boldsymbol{Y}_1) = 1;$ or a miss, which implies the event $\boldsymbol{I}(\boldsymbol{Y}_1) = 0.$ Clearly, if the first shot hits the target, then $\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) = 1$ and we restart the process. However, if the first shot misses the target, then $\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t})$ becomes the probability that a hit occurs in $(\boldsymbol{x}, \boldsymbol{t}]$, or simply $\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) - \boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{x}),$ which is equivalent to $\boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x}).$ Thus, after conditioning on the outcome of the first shot, we write

$$\{\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{Y}_1), \boldsymbol{X}_1^s = \boldsymbol{x}\} = \begin{cases} \boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x}), & \boldsymbol{I}(\boldsymbol{Y}_1) = 0, \\ 1, & \boldsymbol{I}(\boldsymbol{Y}_1) = 1. \end{cases} \tag{17}$$

We then apply the law of total probability over the distribution of $\boldsymbol{I}(\boldsymbol{Y}_1)$, and combine this result with eq 15 to find

$$\begin{aligned} \left[\boldsymbol{M(t)} \mid \boldsymbol{X}_1^s = \boldsymbol{x}\right] &= \int \{\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{Y}_1)\} \boldsymbol{dF}_{\boldsymbol{I}(\boldsymbol{Y}_1)} \\ &= \boldsymbol{p_h} + (1 - \boldsymbol{p_h}) \boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x}). \end{aligned} \tag{18}$$

By combining eqs 16 and 18, we can now describe the state of our process after the first shot

$$\left[\boldsymbol{M(t)} \mid \boldsymbol{X}_1^s = \boldsymbol{x}\right] = \begin{cases} \boldsymbol{p_h} + (1 - \boldsymbol{p_h}) \boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x}), & \boldsymbol{x} \le \boldsymbol{t}, \\ 0, & \boldsymbol{x} > \boldsymbol{t}. \end{cases} \tag{19}$$

Again, by applying the law of total probability, but this time over a range of inter-shot arrival times, we find our functional equivalent for the renewal argument $\boldsymbol{M(t)} = \boldsymbol{P}(\boldsymbol{X}_k^h \le \boldsymbol{t})$ in

$$\begin{aligned} \boldsymbol{M(t)} &= \int_0^\infty \left[\boldsymbol{M(t)} \mid \boldsymbol{X}_1^s = \boldsymbol{x}\right] \boldsymbol{dF}_X^s(\boldsymbol{x}) \\ &= \int_0^t \left[\boldsymbol{p_h} + (1 - \boldsymbol{p_h}) \boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x})\right] \boldsymbol{dF}_X^s(\boldsymbol{x}) \\ &= \boldsymbol{p_h} \boldsymbol{F}_X^s(\boldsymbol{t}) + (1 - \boldsymbol{p_h}) \int_0^t \boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x}) \boldsymbol{dF}_X^s(\boldsymbol{x}). \end{aligned} \tag{20}$$

We leave eq 20 as our mathematical description of the *marksman problem*, and the principal result of this section.

3.2 Generalizing the Marksman Problem

We proceed as in section 3.1 with a combatant and the combatant's unloaded weapon starting a trial at time zero; however, now the combatant must first search the battlefield to find the target. We shall employ the same notation as the previous section by letting $\{S_i^s, I(Y_i)\}$ be our stochastic process. Like the marksman problem, we will model this process as a renewal process in which shots occur at epochs $\{S_i^s\}_{i=1}^{\infty}$, and hits occur at epochs $\{S_k^h\}_{k=1}^{\infty}$. We will let X_i^s, X_k^h, and Y_i take the same meanings as in section 3.1. Y_i will have the same distribution $F_Y(y) = P(Y_i \le y)$, as will X_i^s for $i > 1$ in the distribution $F_X^s(x) = P(X_i^s \le x)$. However, unlike the marksman problem, the first epoch now has the distribution $G_X^s(x) = P(S_1^s \le x) = P(X_1^s \le x)$, and we assume this time represents the time to first detect and engage the target. Karlin and Taylor (1975), Kao (1997), and others term the class of processes represented by this phenomenon *delayed renewal processes*.

Kao (1997) gives the renewal function for a delayed renewal process as

$$M_D(t) = E[N_D(t)] = \sum_{n=1}^{\infty} G(t) * F_{n-1}(t). \tag{21}$$

In eq 21, we employ the subscript D to signify delayed renewal formulation; $N_D(t)$ denotes the delayed counting process; '*' denotes convolution; G and F are, respectively, the distributions of the first and subsequent inter-arrival times; and the subscript n-1 denotes the (n-1)-fold convolution of F (see eq 5). We examine this process by starting with eq 21

$$\begin{aligned} M_D(t) &= \sum_{n=1}^{\infty} G(t) * F_{n-1}(t) \\ &= G(t) * \sum_{m=0}^{\infty} F_m(t) \\ &= G(t) + G(t) * \sum_{m=1}^{\infty} F_m(t) \\ &= G(t) + \int_0^t M(t-x)\,dG(x). \end{aligned} \tag{22}$$

where $M(t) = \sum_{n=1}^{\infty} F_n(t)$. Equation 22 shows that $M_D(t)$ is the solution of a renewal equation of the form

$$M_D(t) = G(t) + \int_0^t M_D(t-x)\,dF(x). \tag{23}$$

One can further show that both $M_D(t)$ and $M(t)$ obey Theorem 1 (Karlin and Taylor, 1975, pp. 198-199), and, in fact, $M_D(t) = M(t)$.

We let the goal for this section be the desire to incorporate the process of target search and detection into the marksman problem; we are now able to reach that goal. First, recall that $\boldsymbol{G}_{\boldsymbol{X}}^{\boldsymbol{s}}(\boldsymbol{t})$ and $\boldsymbol{F}_{\boldsymbol{X}}^{\boldsymbol{s}}(\boldsymbol{t})$ are the respective distributions for the time to fire the first and subsequent shots. Now, invoke the renewal argument $\boldsymbol{M}(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{X}_{\boldsymbol{k}}^{\boldsymbol{h}} \leq \boldsymbol{t})$ to model the time between successive hits. First, let eq 23 take on the quantities $\boldsymbol{G}(\boldsymbol{t}) = \boldsymbol{G}_{\boldsymbol{X}}^{\boldsymbol{s}}(\boldsymbol{t}),\ \boldsymbol{F}(\boldsymbol{t}) = \boldsymbol{F}_{\boldsymbol{X}}^{\boldsymbol{s}}(\boldsymbol{t})$ and $\boldsymbol{M}_{\boldsymbol{D}}(\boldsymbol{t}) = \boldsymbol{M}(\boldsymbol{t})$. Next, compare eq 23 to eq 20. Thus, we immediately find a functional equivalent for our renewal argument in

$$\boldsymbol{M}(\boldsymbol{t}) = \boldsymbol{p}_{\boldsymbol{h}}\boldsymbol{G}_{\boldsymbol{X}}^{\boldsymbol{s}}(\boldsymbol{t}) + (1-\boldsymbol{p}_{\boldsymbol{h}})\int_0^{\boldsymbol{t}} \boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})\,\boldsymbol{dF}_{\boldsymbol{X}}^{\boldsymbol{s}}(\boldsymbol{x}). \tag{24}$$

We leave eq 24 as the mathematical description of the marksman problem with detection and the principal result of this section. However, we now also recognize that the marksman problem of eq 20 is simply a delayed renewal process (as represented by eq 24) with $\boldsymbol{G}(\boldsymbol{t}) = \boldsymbol{F}(\boldsymbol{t})$. We therefore conclude that the delayed renewal process naturally incorporates the marksman problem of Williams and Ancker; hence, we shall refer to the renewal model in eq 24 as the *generalized marksman problem*.

3.3 A Simple Damage Model

In this section, we will again consider a combatant's sequence of shots as well as their outcomes. However, unlike previous sections, we consider the progression of damage inflicted by shots that hit the target. As before, we have a sequence of shots coupled with outcomes, which, if one ignores shots that miss, one can view as a sequence of hits with associated damage outcomes. A possible model for such a sequence is a stochastic process of the form $\{\boldsymbol{S}_{\boldsymbol{k}}^{\boldsymbol{h}}, \boldsymbol{A}_{\boldsymbol{k}}\}$, within which $\boldsymbol{S}^{\boldsymbol{h}}$ would possess a distribution described by the generalized marksman problem (eq 24), and $\boldsymbol{A}$ would denote the damage caused by the hit. While it is possible to view the sequence of shots in such a manner, and sophisticated software codes (BRL-CAD, 2000; MuVES, 2000; ORCA, 2000) exist with which to model the damage a target sustains from a given shot, we desire a simpler analytical model. Hence, we will view our process as $\{\boldsymbol{S}_{\boldsymbol{k}}^{\boldsymbol{h}}, \boldsymbol{I}(\boldsymbol{A}_{\boldsymbol{k}})\}$ and turn to renewal theory for our model.

To begin our investigation of damage effects, let us assume that a given hit either renders the target mission incapable or causes no damage. One might think of this in terms of armor, in that armor either protects absolutely or has no effect. We model this situation via the indicator function $\boldsymbol{I}(\boldsymbol{A}_{\boldsymbol{k}})$ applied in the following sense

$$\boldsymbol{I}(\boldsymbol{A}_{\boldsymbol{k}}) = \begin{cases} 1, & \text{if the hit caused lethal damage,} \\ 0, & \text{if the hit did not cause lethal damage.} \end{cases}$$

Our reliance upon the indicator notation reveals a stochastic process consisting of: (1) a damage process embedded within a (2) hit process. Continuing with the notation of our previous sections, we let the hit events occur at epochs $\{\boldsymbol{S}_{\boldsymbol{k}}^{\boldsymbol{h}}\}_{\boldsymbol{k}=1}^{\infty}$, and define the interval between

consecutive hits to be $X_k^h = S_k^h - S_{k-1}^h$. We introduce the sequence $\{S_l^d\}_{l=1}^{\infty}$, and define it to be the epochs of hits that render the target mission incapable. We will denote the interval between consecutive lethal hits by $X_l^d = S_l^d - S_{l-1}^d$. Figure 4 relates these variables to one another by employing the sample process of figure 1 save only that now we have removed shots that missed the target and placed a diamond on those hits causing lethal damage. Once more, we note that for every S_l^d there is one, and only one S_k^h, such that $S_l^d = S_k^h$.

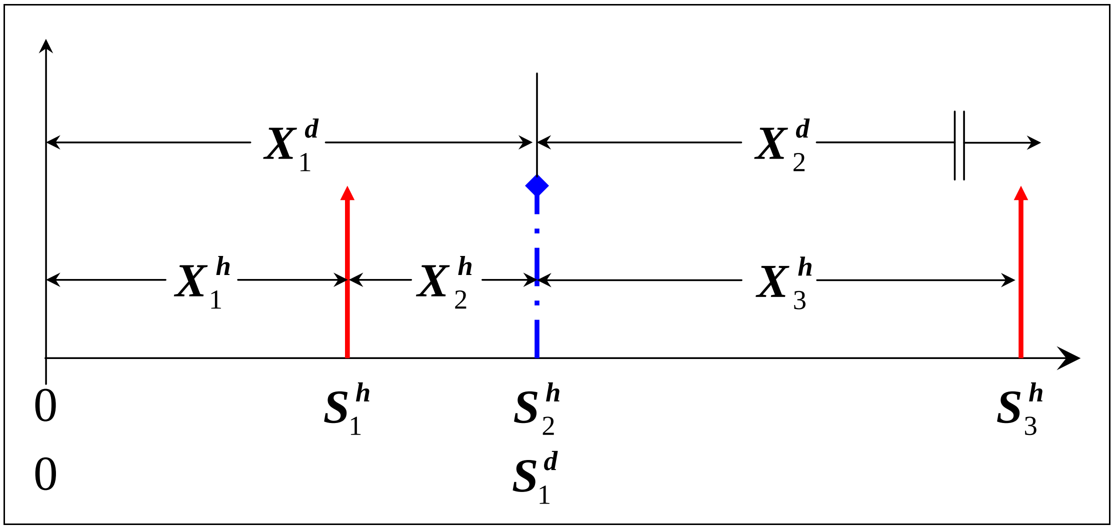

Figure 4. Relationships between key damage model variables.

We assume the intervals X_k^h and damage outcomes A_k to be mutually independent random variables with distributions $F_X^h(t) = P(X_k^h \le t)$ and $F_A(a) = P(A_k \le a)$, respectively. Finally, we turn to Doob (1994) and define the probability of lethal damage (or equivalently the probability of kill) as

$$p_k = \int_{I(A_k)=1} dF_A(a). \tag{25}$$

In addition, we will assume p_k to be known and constant for this paper.

Our goal for this section is to determine the probability distribution of the interval between consecutive lethal hits, and we model this goal via the renewal argument $M(t) = P(X_l^d \le t)$. We begin by conditioning on the time of the first hit

$$\left[M(t) \mid X_1^h = x\right] = \begin{cases} \{P(X_1^d \le t) \mid I(A_1), X_1^h = x\}, & x \le t, \\ 0, & x > t. \end{cases} \tag{26}$$

Again, $\{P(X_1^d \le t) \mid I(A_1), X_1^h = x\}$ reflects that the process depends on the damage caused by the first hit. If the first hit causes lethal damage, then clearly $P(X_1^d \le t) = 1$, and the process restarts. Conversely, if the first hit does not cause damage, then $P(X_1^d \le t)$ becomes the probability that a lethal hit occurs in $(x, t]$, which is simply $P(X_1^d \le t) - P(X_1^d \le x)$, or

$\boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})$. Thus, after conditioning on the outcome of the first hit, we write

$$\{\boldsymbol{P}(\boldsymbol{X}_1^{\boldsymbol{d}} \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{A}_1), \boldsymbol{X}_1^{\boldsymbol{h}} = \boldsymbol{x}\} = \begin{cases} \boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x}), & \boldsymbol{I}(\boldsymbol{A}_1) = 0, \\ 1, & \boldsymbol{I}(\boldsymbol{A}_1) = 1. \end{cases} \tag{27}$$

We now apply the law of total probability, taken over the distribution of $\boldsymbol{I}(\boldsymbol{A}_1)$, to eq 27

$$\begin{aligned} \left[\boldsymbol{M}(\boldsymbol{t}) \mid \boldsymbol{X}_1^{\boldsymbol{h}} = \boldsymbol{x}\right] &= \int \{\boldsymbol{P}(\boldsymbol{X}_1^{\boldsymbol{d}} \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{A}_1), \boldsymbol{X}_1^{\boldsymbol{h}} = \boldsymbol{x}\} \boldsymbol{dF}_{\boldsymbol{I}(\boldsymbol{A}_1)} \\ &= \boldsymbol{p}_{\boldsymbol{k}} + (1-\boldsymbol{p}_{\boldsymbol{k}})\boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x}). \end{aligned} \tag{28}$$

By combining eqs 26 and 28, we can now describe the process after the first shot

$$\left[\boldsymbol{M}(\boldsymbol{t}) \mid \boldsymbol{X}_1^{\boldsymbol{h}} = \boldsymbol{x}\right] = \begin{cases} \boldsymbol{p}_{\boldsymbol{k}} + (1-\boldsymbol{p}_{\boldsymbol{k}})\boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x}), & \boldsymbol{x} \le \boldsymbol{t}, \\ 0, & \boldsymbol{x} > \boldsymbol{t}. \end{cases} \tag{29}$$

Again, by applying the law of total probability, but now over range of inter-shot arrival times, we find

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{t}) &= \int_0^\infty \left[\boldsymbol{M}(\boldsymbol{t}) \mid \boldsymbol{X}_1^{\boldsymbol{h}} = \boldsymbol{x}\right] \boldsymbol{dF}_{\boldsymbol{X}}^{\boldsymbol{h}}(\boldsymbol{x}) \\ &= \int_0^{\boldsymbol{t}} \left[\boldsymbol{p}_{\boldsymbol{k}} + (1-\boldsymbol{p}_{\boldsymbol{k}})\boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})\right] \boldsymbol{dF}_{\boldsymbol{X}}^{\boldsymbol{h}}(\boldsymbol{x}) \\ &= \boldsymbol{p}_{\boldsymbol{k}}\boldsymbol{F}_{\boldsymbol{X}}^{\boldsymbol{h}}(\boldsymbol{t}) + (1-\boldsymbol{p}_{\boldsymbol{k}})\int_0^{\boldsymbol{t}} \boldsymbol{M}(\boldsymbol{t}-\boldsymbol{x})\, \boldsymbol{dF}_{\boldsymbol{X}}^{\boldsymbol{h}}(\boldsymbol{x}). \end{aligned} \tag{30}$$

We leave eq 30 as the primary result of this section, and close by noting that $\boldsymbol{F}_{\boldsymbol{X}}^{\boldsymbol{h}}(\boldsymbol{t})$ may either be a given or obtained from the generalized marksman problem modeled by eq 24. However, we should also note that eq 30 represents a rudimentary, though necessary, first step towards incorporating the effects of damage upon a target. A more detailed, though still elementary, damage model might be a shock model where system failure requires a succession of "shocks" to a system to occur. Both Finkelstein (1996) and Gut (2001) discuss various random shock models. Finkelstein bases his model upon renewal theory, and Gut allows for mixed shocks where several small, or one big shock, can cause failure. While either shock model would likely serve as the next step to a more detailed damage model, for the remainder of this paper, we shall only consider our rudimentary damage model.

4. The Fundamental Duel as a Competing Risk Problem

In the previous section, we focused on a single combatant as he engaged a strictly passive target in order to model the components of a stochastic duel with renewal theory. Now, we will take advantage of this work to model the *fundamental stochastic duel* of Williams and Ancker (1963) as a one-on-one combat between two marksmen we call Red and Blue, a process depicted in figure 5. We form the fundamental duel by assuming Red and Blue continue combat until one is no longer mission capable. This underlying assumption presents us a hidden benefit in our study of that duel: the ability to draw a parallel between the fundamental duel and a "series system." If we equate a combatant that is no longer mission capable to a failed component, we realize that a series system that fails when a component fails models a combat that terminates when a combatant becomes mission incapable. When we view a system, and therefore the fundamental duel, in this manner, we recognize a specific instance of a general class of problems that David and Moeschberger (1978) term *competing risk problems*. Our goal is to employ the method of competing risks to model the fundamental duel, a goal we will attain by undertaking three tasks:

1. Express the time to system failure as a function of the components.
2. Express the probability that one component fails before the remaining components.
3. Relate these expressions to the fundamental duel.

We will address the first two tasks as we introduce the general methods of competing risks, and arrive at the third goal as we close the section.

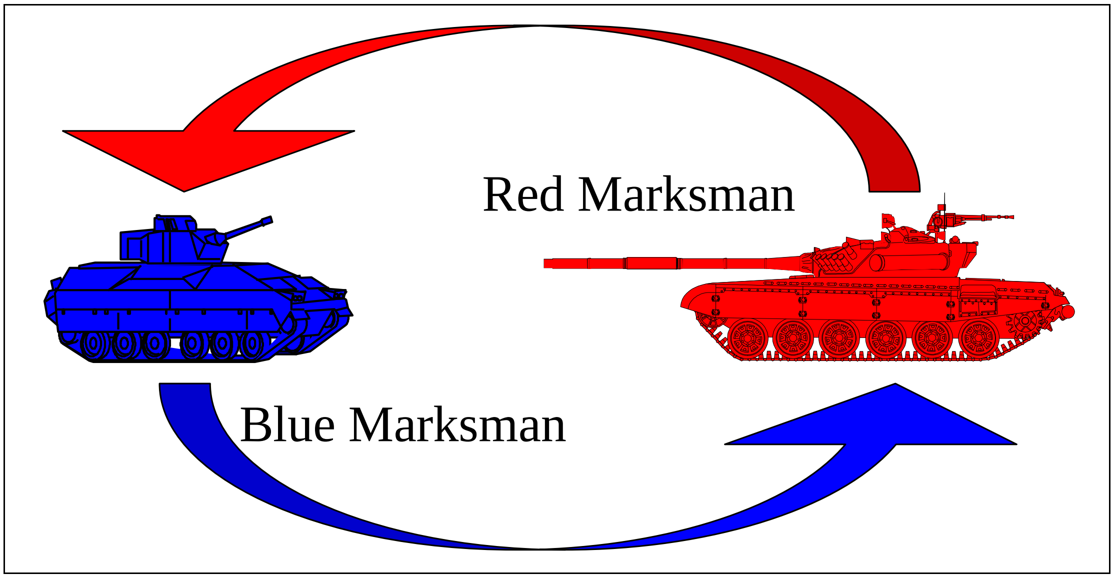

Figure 5. The fundamental duel as two marksmen.

We start with a system that consists of $k > 1$ components without imposing a relationship among these components. We further assume that the likelihood of two or more component failures in the same instant of time to be negligible. David and Moeschberger (1978) state that, in general, such a system possesses two observable properties: (1) the time when failure occurs, and (2) which of the k components failed. If we let $\{L_i\}_{i=1}^{k}$ denote the unobservable lifetimes for the respective components, we can then define the system lifetime L in terms of these lifetimes by letting $L = \min\{L_1, L_2, \ldots, L_k\}$. Undoubtedly, if L exceeds some value t, then each L_i must also exceed t; thus, David and Moeschberger write

$$\begin{aligned} P(L > t) &= P(L_1 > t, L_2 > t, \ldots, L_k > t) \\ &= 1 - F_L(t) \\ &= F_L^s(t), \end{aligned} \tag{31}$$

where $F_L^s(t)$ is often termed the survivor function. David and Moeschberger define the *conditional failure rate* for the system as

$$r_L(t) = \frac{f_L(t)}{1 - F_L(t)} = \frac{f_L(t)}{F_L^s(t)}. \tag{32}$$

They further interpret $r_L(t)$ to mean the probability that a system, given that it is operational at time t, will fail in $(t, t + dt)$, owing to a failure of any of its components. One can also define the conditional failure rate for the system in terms of the failure rate for each component. First, let $g_i(t)dt$ for $i = 1, 2, \cdots, k$ denote the probability that a system, given each component operational at t, fails in $(t, t + dt)$ because component i failed. Accordingly, David and Moeschberger determine that

$$r_L(t) = \sum_{i=1}^{k} g_i(t). \tag{33}$$

Furthermore, they infer from eq 33 that one can view the "health" of the system as a function of the health of each component (1978, p. 4).

We are now ready to address the first two goals we set at the beginning of this section. We begin by rewriting eq 32 as

$$r_L(t) = -\frac{d}{dt} \ln F_L^s(t), \tag{34}$$

then, from David and Moeschberger, we determine that

$$Q(a, t) = \exp\left[-\int_a^t r_L(\tau)\, d\tau \right]. \tag{35}$$

Equation 35 defines the probability that the system, having been operational at $\boldsymbol{a}$, remains so at $\boldsymbol{a} + \boldsymbol{t}$. With this result, David and Moeschberger subsequently determine the probability that a system fails in $(\boldsymbol{a}, \boldsymbol{t})$ due to a failure of component $\boldsymbol{i}$

$$\boldsymbol{Q}_i(\boldsymbol{a}, \boldsymbol{t}) = \int_{\boldsymbol{a}}^{\boldsymbol{t}} \boldsymbol{g}_i(\tau)\boldsymbol{Q}(\boldsymbol{a}, \tau)\, \boldsymbol{d}\tau. \tag{36}$$

Equations 35 and 36 respectively satisfy the requirements of our first two tasks. However, we now note that while computing $\boldsymbol{r}_L(\boldsymbol{t})$ for a general system is often clear-cut, computing $\boldsymbol{g}_i(\boldsymbol{t})$ for each component of that system calls for some knowledge of the system.

Until this point in our discussion, nothing has required us to impose a relationship upon the component lifetimes. The literature on competing risks encompasses a large amount of work on models where the relationships among lifetimes are independent, dependent, or some combination of both. In addition, this body of literature also provides substantial discussion on the problem of identifying lifetime models with particular functional relationships among the component lifetimes. We, including other researchers such as Williams and Ancker (1963), assumed the relationship between a marksman and his target to be a passive relationship. In this sense, a passive target is one that is unaffected by the act of having been fired upon, but as Bathe and McNaught (1989) observed, this assumption is questionable. Still, in order to reach our immediate goal, we now find it necessary to rely upon a passive target. For the moment, we shall justify this assumption by noting that Tsiatis (1975), Peterson (1976), and Miller (1977) independently suggest that, absent specific information regarding the structure of the system, without an assumption of independence among the system components, we cannot determine the required joint or marginal distributions for the component lifetimes. However, we contend that by assuming a passive target, one implies a relationship between a marksman and his target, which, in the fundamental duel, naturally leads to the assumption of independence. Nevertheless, as we conclude this paper, we shall have available a means to construct a framework that will allow alternative structural relationships between the combatants in a fundamental duel; thus, we believe that further research will allow us to relax the requirement of independence.

Therefore, by assuming independence among the subsystem lifetimes, eq 31 becomes

$$\boldsymbol{P}(\boldsymbol{L} > \boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{L}_1 > \boldsymbol{t})\boldsymbol{P}(\boldsymbol{L}_2 > \boldsymbol{t}) \cdots \boldsymbol{P}(\boldsymbol{L}_k > \boldsymbol{t}),$$

or equivalently

$$\boldsymbol{F}_L^s(\boldsymbol{t}) = \prod_{i=1}^{k} \boldsymbol{F}_i^s(\boldsymbol{t}), \tag{37}$$

where $\boldsymbol{F}_i^s(\boldsymbol{t}) = 1 - \boldsymbol{F}_i(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{L}_i > \boldsymbol{t}).$ We also have

$$\boldsymbol{g}_i(\boldsymbol{t}) = \frac{\boldsymbol{f}_i(\boldsymbol{t})}{\boldsymbol{F}_L^s(\boldsymbol{t})} \prod_{\substack{j=1 \\ j \neq i}}^{k} \boldsymbol{F}_j^s(\boldsymbol{t}) = \frac{\boldsymbol{f}_i(\boldsymbol{t})}{\boldsymbol{F}_i^s(\boldsymbol{t})}, \tag{38}$$

where $\boldsymbol{f}_i(\boldsymbol{t})$ is the density of $\boldsymbol{F}_i(\boldsymbol{t})$. David and Moeschberger (1978) equate the last quantity in eq 38 to $\boldsymbol{r}_i(\boldsymbol{t})$, and term it the *cause specific failure rate*, or the *marginal intensity function*. Thus, for a system with independent component lifetimes, they establish that

$$\boldsymbol{g}_i(\boldsymbol{t}) = \boldsymbol{r}_i(\boldsymbol{t}) \quad \boldsymbol{i} = 1, 2, \cdots, \boldsymbol{k}. \tag{39}$$

David and Moeschberger also state that for independent component lifetimes, the probability of system failure because a given component fails in $(\boldsymbol{t}, \boldsymbol{t} + \boldsymbol{dt})$ is the same whether that component is one of $\boldsymbol{k}$ or the only component present. Consequently, by assuming independence, eqs 35 and 36 respectively become

$$\boldsymbol{Q}(\boldsymbol{a}, \boldsymbol{t}) = \exp\left[-\int_{\boldsymbol{a}}^{\boldsymbol{t}} \sum_{\boldsymbol{i}=1}^{\boldsymbol{k}} \boldsymbol{r}_i(\tau)\, \boldsymbol{d}\tau\right] = \exp\left[-\sum_{\boldsymbol{i}=1}^{\boldsymbol{k}} \int_{\boldsymbol{a}}^{\boldsymbol{t}} \boldsymbol{r}_i(\tau)\, \boldsymbol{d}\tau\right], \tag{40}$$

and

$$\boldsymbol{Q}_i(\boldsymbol{a}, \boldsymbol{t}) = \int_{\boldsymbol{a}}^{\boldsymbol{t}} \boldsymbol{r}_i(\tau) \boldsymbol{Q}(\boldsymbol{a}, \tau)\, \boldsymbol{d}\tau. \tag{41}$$

With eq 41, we can now define the fundamental duel as a competing risk. Let us define the events $\boldsymbol{R}$ and $\boldsymbol{B}$, and take them to signify, respectively, that Red or Blue was victorious in the combat. Furthermore, assume the probability of Red and Blue simultaneously rendering each-other mission incapable to be negligible; thus $\boldsymbol{P}(\boldsymbol{R}) + \boldsymbol{P}(\boldsymbol{B}) = 1$. We signify that our fundamental duel is a one-on-one engagement when we set $\boldsymbol{k} = 2$, and thus from eq 41, we define

$$\begin{aligned} \boldsymbol{P}(\boldsymbol{B}) &\equiv \boldsymbol{Q}_1(0, \infty) = \int_0^\infty \boldsymbol{r}_1(\tau) \boldsymbol{Q}(0, \tau)\, \boldsymbol{d}\tau \\ \boldsymbol{P}(\boldsymbol{R}) &\equiv \boldsymbol{Q}_2(0, \infty) = \int_0^\infty \boldsymbol{r}_2(\tau) \boldsymbol{Q}(0, \tau)\, \boldsymbol{d}\tau. \end{aligned} \tag{42}$$

In their original formulation of the fundamental duel, Williams and Ancker (1963) set the condition for victory to be the first combatant to score a hit upon the other. We can model this situation by computing $\boldsymbol{r}_1(\boldsymbol{t})$ and $\boldsymbol{r}_2(\boldsymbol{t})$ from the requisite distributions. The simplest method is via the generalized marksman problem (eq 24), with the appropriate data for Red and Blue; however, we could also set $\boldsymbol{p}_k = 1$ and resort to the damage model (eq 30) as an alternative.

Equation 42 completes our work for this section; however, obtaining useful analytical results with a competing risk formulation can prove difficult. In the following section, we consider two examples that Williams and Ancker originally studied (1963). We shall see that in the first example, a competing risk formulation is an asset, while in the second it is a drawback. Although in the latter case, we present an alternative approach that yields a solution equivalent to that obtained by Williams and Ancker.

5. Analytical Validation of the Renewal Duel Formulation

In the previous section, we constructed the fundamental duel as a competing risk problem; we now evaluate that approach by studying the same examples that Williams and Ancker considered (1963). As in earlier sections, we make use of two combatants, Red and Blue, engaged in a one-on-one combat; we declare the victor to be the first combatant to score a hit upon the other. Williams and Ancker obtained results for such a combat that correspond to the probability of Blue surviving engagement with Red in each of their original examples. Our goal is to represent these probabilities by employing a competing risk framework. To reach this goal, we require the marginal intensity functions for Red and Blue. In order to find these functions, however, we must know how the interval between consecutive hits (not shots) is distributed. If $\boldsymbol{X_B^h}$ and $\boldsymbol{X_R^h}$ describe the interval between consecutive hits, then we must find $\boldsymbol{F_B^h(t) = P(X_B^h \le t)}$ or $\boldsymbol{F_R^h(t) = P(X_R^h \le t)}$ as necessary. However, since the mathematical models describing the Red and Blue marksman processes are the same, we need only derive the marginal intensity function for Blue, and then to obtain Red's we simply replace the appropriate subscripts. In the remainder of this section, we employ $\boldsymbol{p_R}$ and $\boldsymbol{p_B}$ to denote the respective constant probabilities of hit for Red and Blue. In addition, we will let $\boldsymbol{f_B^h(t)}$ and $\boldsymbol{f_R^h(t)}$ respectively denote the densities of $\boldsymbol{F_B^h(t)}$ and $\boldsymbol{F_R^h(t)}$.

We first consider a case where the interval between consecutive shots taken by Blue has the exponential probability density $\boldsymbol{f_B^s(t) = \lambda_B \exp(-\lambda_B t)}$, with $\boldsymbol{\varphi_B^s(s) = \lambda_B / (s + \lambda_B)}$ as the Laplace transform of $\boldsymbol{f_B^s(t)}$. Let $\boldsymbol{\Phi_B^h(s)}$ denote the Laplace transform of $\boldsymbol{f_B^h(t)}$. We obtain $\boldsymbol{\Phi_B^h(s)}$ by differentiating eq 20 with respect to $\boldsymbol{t}$, taking the Laplace transform of the result, and rearranging terms. Thus, we write

$$\boldsymbol{\Phi_B^h(s)} = \frac{\boldsymbol{p_B \varphi_B^s(s)}}{1-(1-\boldsymbol{p_B})\boldsymbol{\varphi_B^s(s)}} = \frac{\boldsymbol{p_B \lambda_B}}{\boldsymbol{s + p_B \lambda_B}}. \tag{43}$$

By inverting eq 43, we obtain

$$\boldsymbol{f_B^h(t) = p_B \lambda_B} \exp(-\boldsymbol{p_B \lambda_B t}), \tag{44}$$

which we then integrate to find

$$\boldsymbol{F_B^h(t)} = \int_0^{\boldsymbol{t}} \boldsymbol{f_B^h(t)\,dt} = 1 - \exp(-\boldsymbol{p_B \lambda_B}). \tag{45}$$

We can now obtain the marginal intensity function for Blue by turning to eq 38

$$\boldsymbol{r_B}(\boldsymbol{t}) = \frac{\boldsymbol{f_B^h}(\boldsymbol{t})}{1-\boldsymbol{F_B^h}(\boldsymbol{t})} = \frac{\boldsymbol{p_B}\boldsymbol{\lambda_B}\exp(-\boldsymbol{p_B}\boldsymbol{\lambda_B})}{1-\left[1-\exp(-\boldsymbol{p_B}\boldsymbol{\lambda_B})\right]} = \boldsymbol{p_B}\boldsymbol{\lambda_B}. \tag{46}$$

We find $\boldsymbol{r_R}(\boldsymbol{t})$ by replacing the subscript ***B*** with ***R*** in eq 46. By turning to eq 40, we compute the time until combat termination

$$\boldsymbol{Q}(0,\boldsymbol{t}) = \exp\left[-\int_0^{\boldsymbol{t}} \boldsymbol{r_B}(\tau)\,\boldsymbol{d}\tau - \int_0^{\boldsymbol{t}} \boldsymbol{r_R}(\tau)\,\boldsymbol{d}\tau\right] = \exp\left[-(\boldsymbol{p_B}\boldsymbol{\lambda_B} + \boldsymbol{p_R}\boldsymbol{\lambda_R})\boldsymbol{t}\right]. \tag{47}$$

If we now substitute the quantities given by eqs 46 and 47 into eq 42, we obtain our desired probability of survival

$$\begin{aligned}\boldsymbol{P}(\boldsymbol{B}) \equiv \boldsymbol{Q}_1(0,\infty) &= \int_0^{\infty} \boldsymbol{r_B}(\tau)\,\boldsymbol{Q}(0,\tau)\,\boldsymbol{d}\tau \\ &= \int_0^{\infty} \boldsymbol{p_B}\boldsymbol{\lambda_B}\exp\left[-(\boldsymbol{p_B}\boldsymbol{\lambda_B} + \boldsymbol{p_R}\boldsymbol{\lambda_R})\tau\right]\boldsymbol{d}\tau \\ &= \frac{\boldsymbol{p_B}\boldsymbol{\lambda_B}}{\boldsymbol{p_B}\boldsymbol{\lambda_B} + \boldsymbol{p_R}\boldsymbol{\lambda_R}}.\end{aligned} \tag{48}$$

Thus, we find that, in this case, a competing risk framework yields the same result as Williams and Ancker (1963, eq 6).

Consider the second example of Williams and Ancker, a case where the interval between consecutive shots taken by the combatants has an Erlang(2) distribution. We describe this distribution and its associated Laplace transform by

$$\begin{aligned}&\boldsymbol{f_B^s}(\boldsymbol{t}) = \boldsymbol{\lambda_B^2}\boldsymbol{t}\exp(-\boldsymbol{\lambda_B}\boldsymbol{t}), \quad \boldsymbol{t} \ge 0, \boldsymbol{\lambda_B} > 0 \\ &\boldsymbol{\varphi_B^s}(\boldsymbol{s}) = \left(\frac{\boldsymbol{\lambda_B}}{\boldsymbol{s}+\boldsymbol{\lambda_B}}\right)^2.\end{aligned} \tag{49}$$

Recall that $\boldsymbol{\Phi_B^h}(\boldsymbol{s})$ denotes the Laplace transform of $\boldsymbol{f_B^h}(\boldsymbol{t})$. Once again, we find this quantity by differentiating eq 20 with respect to ***t***, taking the Laplace transform of the result, and rearranging terms. Thus, we write

$$\boldsymbol{\Phi_B^h}(\boldsymbol{s}) = \frac{\boldsymbol{p_B}\boldsymbol{\varphi_B^s}(\boldsymbol{s})}{1-(1-\boldsymbol{p_B})\boldsymbol{\varphi_B^s}(\boldsymbol{s})} = \frac{\boldsymbol{p_B}\boldsymbol{\lambda_B^2}}{\boldsymbol{s}^2 + 2\boldsymbol{\lambda_B}\boldsymbol{s} + \boldsymbol{p_B}\boldsymbol{\lambda_B^2}} = \frac{\boldsymbol{p_B}\boldsymbol{\lambda_B^2}}{(\boldsymbol{s}+\boldsymbol{a_B})(\boldsymbol{s}+\boldsymbol{b_B})}, \tag{50}$$

where we have let $\boldsymbol{a_B} = \boldsymbol{\lambda_B}(1+\sqrt{1-\boldsymbol{p_B}})$, and $\boldsymbol{b_B} = \boldsymbol{\lambda_B}(1-\sqrt{1-\boldsymbol{p_B}})$. By inverting eq 50, we obtain

$$\boldsymbol{f_B^h}(\boldsymbol{t}) = \left(\frac{\boldsymbol{p_B}\boldsymbol{\lambda_B^2}}{\boldsymbol{b_B}-\boldsymbol{a_B}}\right)\left\{\exp(-\boldsymbol{a_B}\boldsymbol{t}) - \exp(-\boldsymbol{b_B}\boldsymbol{t})\right\}. \tag{51}$$

We then integrate eq 51 to find

$$\begin{aligned}
\boldsymbol{F}_{\boldsymbol{B}}^{\boldsymbol{h}}(\boldsymbol{t}) &= \int_0^{\boldsymbol{t}} \boldsymbol{f}_{\boldsymbol{B}}^{\boldsymbol{h}}(\boldsymbol{t})\boldsymbol{dt} \\
&= \left(\frac{\boldsymbol{p}_{\boldsymbol{B}}\lambda_{\boldsymbol{B}}^2}{\boldsymbol{b}_{\boldsymbol{B}}-\boldsymbol{a}_{\boldsymbol{B}}}\right)\left\{\frac{\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})}{\boldsymbol{b}_{\boldsymbol{B}}}-\frac{\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})}{\boldsymbol{a}_{\boldsymbol{B}}}-\frac{1}{\boldsymbol{b}_{\boldsymbol{B}}}+\frac{1}{\boldsymbol{a}_{\boldsymbol{B}}}\right\} \qquad (52) \\
&= 1-\frac{\boldsymbol{b}_{\boldsymbol{B}}\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\boldsymbol{a}_{\boldsymbol{B}}\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})}{\boldsymbol{b}_{\boldsymbol{B}}-\boldsymbol{a}_{\boldsymbol{B}}}.
\end{aligned}$$

We now obtain the marginal intensity function for Blue

$$\begin{aligned}
\boldsymbol{r}_{\boldsymbol{B}}(\boldsymbol{t}) &= \frac{\boldsymbol{f}_{\boldsymbol{B}}^{\boldsymbol{h}}(\boldsymbol{t})}{1-\boldsymbol{F}_{\boldsymbol{B}}^{\boldsymbol{h}}(\boldsymbol{t})} \\
&= \frac{\left(\frac{\boldsymbol{p}_{\boldsymbol{B}}\lambda_{\boldsymbol{B}}^2}{\boldsymbol{b}_{\boldsymbol{B}}-\boldsymbol{a}_{\boldsymbol{B}}}\right)\left\{\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})\right\}}{1-\left[1-\frac{\boldsymbol{b}_{\boldsymbol{B}}\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\boldsymbol{a}_{\boldsymbol{B}}\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})}{\boldsymbol{b}_{\boldsymbol{B}}-\boldsymbol{a}_{\boldsymbol{B}}}\right]} \\
&= \frac{\left(\frac{\boldsymbol{p}_{\boldsymbol{B}}\lambda_{\boldsymbol{B}}^2}{\boldsymbol{b}_{\boldsymbol{B}}-\boldsymbol{a}_{\boldsymbol{B}}}\right)\left\{\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})\right\}}{\frac{\boldsymbol{b}_{\boldsymbol{B}}\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\boldsymbol{a}_{\boldsymbol{B}}\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})}{\boldsymbol{b}_{\boldsymbol{B}}-\boldsymbol{a}_{\boldsymbol{B}}}} \qquad (53) \\
&= \frac{\boldsymbol{p}_{\boldsymbol{B}}\lambda_{\boldsymbol{B}}^2\left\{\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})\right\}}{\boldsymbol{b}_{\boldsymbol{B}}\exp(-\boldsymbol{a}_{\boldsymbol{B}}\boldsymbol{t})-\boldsymbol{a}_{\boldsymbol{B}}\exp(-\boldsymbol{b}_{\boldsymbol{B}}\boldsymbol{t})}.
\end{aligned}$$

We find $\boldsymbol{r}_{\boldsymbol{R}}(\boldsymbol{t})$ by replacing the subscript ***B*** with ***R*** in eq 53. By turning to eq 40, we now compute the time until combat termination

$$\boldsymbol{Q}(0,\boldsymbol{t}) = \exp\left[-\int_0^{\boldsymbol{t}}\boldsymbol{r}_{\boldsymbol{B}}(\tau)\boldsymbol{d}\tau-\int_0^{\boldsymbol{t}}\boldsymbol{r}_{\boldsymbol{R}}(\tau)\boldsymbol{d}\tau\right]. \qquad (54)$$

If we substituted the quantities given by eqs 53 and 54 into eq 42

$$\boldsymbol{P}(\boldsymbol{B}) \equiv \int_0^{\infty}\boldsymbol{r}_{\boldsymbol{B}}(\tau)\boldsymbol{Q}(0,\tau)\boldsymbol{d}\tau \qquad (55)$$

and then carried out all of the integrations, we would obtain the probability of survival we sought; however, that integration is extremely difficult. In this case, we see a competing risk formulation that leads to a very complicated solution, but we have other means available to find an analytical solution for our Erlang(2) case.

Because we employed the criterion that the first combatant to score a hit upon the other becomes the victor, we can take another path to obtain a competing risk solution to the Erlang(2) case. Let $\boldsymbol{X}_{\boldsymbol{B}}^{\boldsymbol{h}}$ and $\boldsymbol{X}_{\boldsymbol{R}}^{\boldsymbol{h}}$ denote the respective lifetimes of Blue and Red respectively; thus, $\boldsymbol{Z} = \min\left\{\boldsymbol{X}_{\boldsymbol{B}}^{\boldsymbol{h}}, \boldsymbol{X}_{\boldsymbol{R}}^{\boldsymbol{h}}\right\}$ defines the length of the combat. If we let the indicator function, $\boldsymbol{I}$, take on the values

$$\boldsymbol{I} = \begin{cases} 1, & \text{If Blue hits Red before Red hits Blue,} \\ 0, & \text{If Red hits Blue before Blue hits Red,} \end{cases}$$

then the indicator function describes the victor. Let the event, $\boldsymbol{B}$, denote that Blue is victorious; thus, we can define

$$\boldsymbol{P}(\boldsymbol{B}) \equiv \boldsymbol{P}(\boldsymbol{I} = 1) = \boldsymbol{P}(\boldsymbol{X}_{\boldsymbol{B}}^{\boldsymbol{h}} < \boldsymbol{X}_{\boldsymbol{R}}^{\boldsymbol{h}}). \tag{56}$$

We have available an expression for $\boldsymbol{f}_{\boldsymbol{B}}^{\boldsymbol{h}}(\boldsymbol{t})$ in eq 51, and by a change of subscripts, we also have available an expression for $\boldsymbol{f}_{\boldsymbol{R}}^{\boldsymbol{h}}(\boldsymbol{t})$; therefore, we can immediately compute $\boldsymbol{P}(\boldsymbol{B})$ by

$$\begin{aligned}
\boldsymbol{P}(\boldsymbol{B}) &\equiv \boldsymbol{P}(\boldsymbol{X}_{\boldsymbol{B}}^{\boldsymbol{h}} < \boldsymbol{X}_{\boldsymbol{R}}^{\boldsymbol{h}}) \\
&= \int_0^\infty \boldsymbol{f}_{\boldsymbol{B}}^{\boldsymbol{h}}(\boldsymbol{t}) \int_{\boldsymbol{t}}^\infty \boldsymbol{f}_{\boldsymbol{R}}^{\boldsymbol{h}}(\tau)\, \boldsymbol{d}\tau\, \boldsymbol{dt} \\
&= \left(\frac{\boldsymbol{p}_{\boldsymbol{B}}\lambda_{\boldsymbol{B}}^2}{\boldsymbol{b}_{\boldsymbol{B}} - \boldsymbol{a}_{\boldsymbol{B}}}\right)\left(\frac{\boldsymbol{p}_{\boldsymbol{R}}\lambda_{\boldsymbol{R}}^2}{\boldsymbol{b}_{\boldsymbol{R}} - \boldsymbol{a}_{\boldsymbol{R}}}\right)\left\{\begin{matrix} \dfrac{1}{(\boldsymbol{a}_{\boldsymbol{B}} + \boldsymbol{a}_{\boldsymbol{R}})\boldsymbol{a}_{\boldsymbol{R}}} - \dfrac{1}{(\boldsymbol{b}_{\boldsymbol{B}} + \boldsymbol{a}_{\boldsymbol{R}})\boldsymbol{a}_{\boldsymbol{R}}} - \\ \dfrac{1}{(\boldsymbol{a}_{\boldsymbol{B}} + \boldsymbol{b}_{\boldsymbol{R}})\boldsymbol{b}_{\boldsymbol{R}}} + \dfrac{1}{(\boldsymbol{b}_{\boldsymbol{B}} + \boldsymbol{b}_{\boldsymbol{R}})\boldsymbol{b}_{\boldsymbol{R}}} \end{matrix}\right\}.
\end{aligned} \tag{57}$$

In eq 57, $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same meanings as for eq 51. After considerable algebraic manipulation eq 57 reduces to the same result as Williams and Ancker obtained for the Erlang(2) case (1963, eq 24).

When we couple this section with the previous one, we conclude that we can construct the fundamental duel using a competing risk framework. However, we also realize that a competing risk framework does not necessarily lead to the nice analytical solutions that Williams and Ancker obtained in their studies. Consequently, we ask ourselves what benefits do we accrue by employing competing risks in our study of stochastic duels, and we respond by highlighting the work of Ewing et al. (2002) who modeled the complex interactions of a pest with a California citrus crop. In their model, Ewing et al. assumed a hazard function that was reasonably well behaved over time (2002, p. 41); as a result, they described the occurrence of events in an ecosystem via this hazard function. We close by noting that while a competing risk framework may not offer a nice analytical solution, the framework does offer the potential to assist in computer simulation studies.

6. Stochastic Duels and Layered Survivability

In this section, we introduce the concept of *layered survivability* (Horton, 1996) in order to build the foundation for the remainder of this paper. To reach this goal, we undertake two tasks. First, we relate both the generalized marksman problem and the damage process to the encounter process that underpins layered survivability. Second, we utilize this concept as a tool with which to modify the generalized marksman and damage processes so they include appropriate survivability responses. To aid our discussion, figure 6 reproduces the survivability "onion" diagram that describes layered survivability. We note that layered survivability models an encounter between combatants who can interact, but whose opportunity for interaction is not certain. From the discussion thus far, we now know that we can fill the role of encounter with the fundamental duel as illustrated by figure 5.

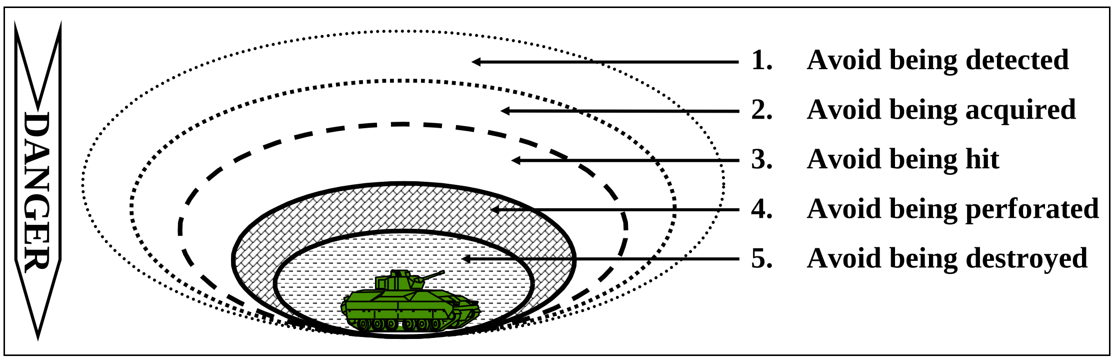

Figure 6. The layers of survivability (Horton, 1996).

First, we draw attention to the clear and the filled areas of figure 6. Observe that the boundary between these areas represents a physical break in an encounter; in a manner of speaking, the boundary represents the "skin" of the platform. If one thinks of the vertical dimension in figure 6 as a measure of time that increases as we move closer to the target, then above the fourth layer, an adversary has yet to hit; hence, we may only consider the likelihood of that event by modeling it probabilistically. However, by passing into the fourth layer, the adversary has hit almost surely; hence, we must deal with the damage rendered by that hit, damage that can range from instant, catastrophic failure to no damage at all. Here, we model events before the fourth layer via the generalized marksman process. Necessarily, the outcome of the generalized marksman process drives our need to evaluate the damage a target sustains; hence, we model events in and beyond the fourth ring via our damage model. Accordingly, figure 7 illustrates how these processes are coupled, in the context of the fundamental duel described by figure 5, in order to capture the essence of layered survivability; the processes in italics represent one half of the encounter and the non-italicized processes the other half.

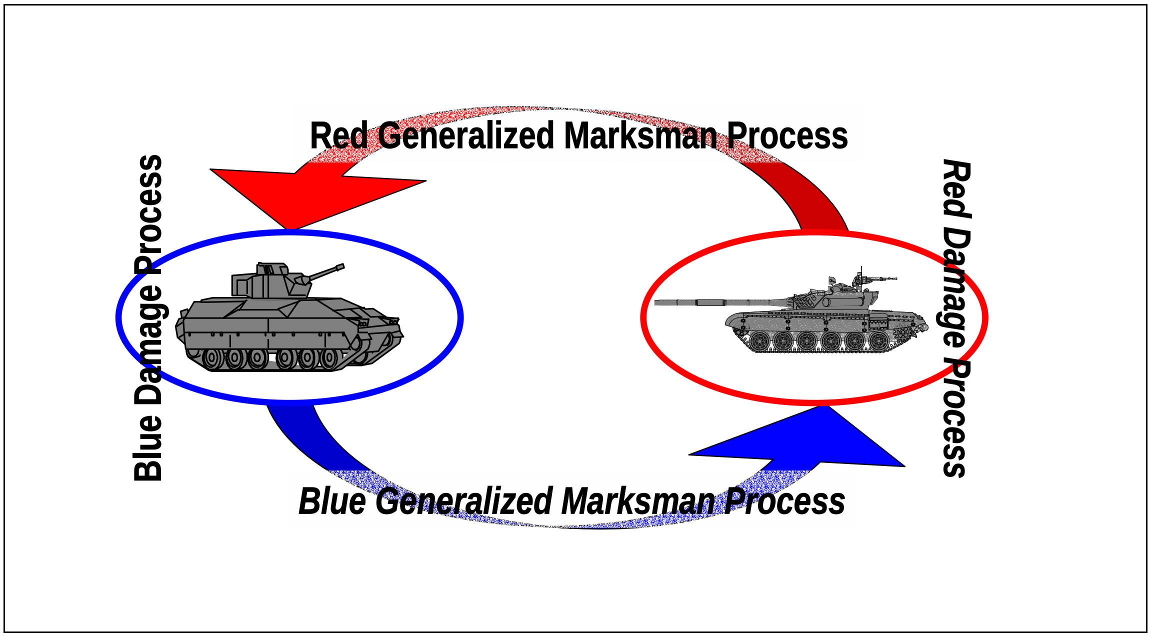

Figure 7. The fundamental duel decomposed.

For purposes of this section, we have completed our first task. However, as we discuss figure 6, we should resist the temptation to characterize each layer as a binary event. With respect to a platform and its crew, only two layers truly represent events that alter the crews' perception of the platforms' *state of nature*: detection and destruction. For detection, the crew either knows with certainty that there is an adversary in proximity, or they do not. If the crew chooses to engage the adversary, they either know with certainty that they have defeated the adversary, or they have not. Thus, in the fundamental duel, the intervening layers are the exemplar for the *wrestling match* that Clausewitz described in his quotation regarding war:

> War is nothing but a duel on a larger scale. Countless duels go to make up a war, but a picture of it as a whole can be formed by imagining a pair of wrestlers. Each tries through physical force to compel the other to do his will; his *immediate* aim is to *throw* his opponent in order to make him incapable of further resistance. (1976, p. 75)

We suggest that survivability exhibits a dual nature, one that occurs because a platform both fights and defends during a combat; how well it survived was a function of successes in both facets. Figure 7 illustrates this duality. From the point of view of the platform, defense responds to the arrow directed at the platform, which we typically refer to as platform survivability. Attack is represented by the arrow directed at the opposing platform, which is platform lethality.

Now, let us focus on platform survivability and for ease of discussion, we can think of this in terms of the Red generalized marksman process coupled to a Blue damage process. The processes of detection and hit are intuitive and directly considered in the marksman process; however, we must choose how to handle acquisition. In some cases, we may model acquisition

as part of the time to detect; while in other cases, we may model it as part of the inter-firing time. Consider a turreted platform that has detected a target. In this case, the acquisition time would represent the time to slew the turret to aim at the target; hence, we would model the time to slew the turret as part of the time to detect and fire the first shot. In another example, consider a platform that must reacquire after each shot. In this case, we would choose to model the time to reacquire as part of the inter-firing time. Now, given our rudimentary damage model, we simply choose to consider the probability of lethal damage to represent the effect of armor. If the armor protects us from damage, we survive that hit; however, if the armor fails, then we do not survive. A more detailed damage model will require changes, but for this discussion, this notion is sufficient. Thus, we have related our work in this paper to the concept of layered survivability, and completed our second task. Let us now add survivability measures to our renewal duel formulation.

7. Stochastic Duels: Adding Survivability Measures

In this section, we again study a one-on-one engagement between a combatant and his target; however, since we know that the marksman problem of Williams and Ancker is a special case of our generalized marksman, we need not parallel the development in section 3. First, we begin with the combatant's generalized marksman process, and add survivability measures that will disrupt his firing process. Next, we continue with the target's damage process, and add survivability measures that perturb the interaction of the combatant's projectile and the targets armor. Finally, at the conclusion of this section, we will have a complete renewal duel formulation with survivability with which to model the fundamental duel of Williams and Ancker.

7.1 The Generalized Marksman with Survivability

We begin by considering a combatant as he fires at a target. We have modeled this firing process via the generalized marksman problem (eq 24). Recall that the combatant begins with an unloaded weapon at time zero. He must also search the battlefield to find a target; however, now the target is no longer passive, as we will allow the target to perturb the combatant's firing process. We let $\{\boldsymbol{S}_i^s, \boldsymbol{I}(\boldsymbol{Y}_i), \boldsymbol{I}(\boldsymbol{W}_i)\}$ denote our stochastic process; however, we augmented the notation of section 3.2 by introducing $\boldsymbol{W}$ to reflect that the process also depends on the survivability response. We employ the indicator function in the following sense

$$\boldsymbol{I}(\boldsymbol{W}_i) = \begin{cases} 1, & \text{the survivability measure perturbed the shot,} \\ 0, & \text{the survivability measure failed to perturb the shot.} \end{cases}$$

As before, we employ renewal theory to model this process in which shot epochs occur $\{\boldsymbol{S}_i^s\}_{i=1}^{\infty}$, and hits occur at epochs $\{\boldsymbol{S}_k^h\}_{k=1}^{\infty}$. $\boldsymbol{X}_i^s$, $\boldsymbol{X}_k^h$, and $\boldsymbol{Y}_i$ will take the same meanings as section 3.2. $\boldsymbol{X}_i^s$ is distributed as

$$\boldsymbol{P}(\boldsymbol{X}_i^s \le \boldsymbol{x}) = \begin{cases} \boldsymbol{F}_X^s(\boldsymbol{x}), & \boldsymbol{i} > 1, \\ \boldsymbol{G}_X^s(\boldsymbol{x}), & \boldsymbol{i} = 1. \end{cases}$$

Obviously, the shot outcome, $\boldsymbol{Y}_i$, will depend on the survivability response. While for most practical survivability measures one should not assume $\boldsymbol{Y}_i$ and $\boldsymbol{X}_i^s$ to be stochastically independent, we will start by making this assumption.

We begin by invoking the renewal argument that $\boldsymbol{M}(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{X}_i^h \le \boldsymbol{t})$, and proceed by conditioning on the time of arrival of the first shot

$$\left[\boldsymbol{M}(\boldsymbol{t}) \mid \boldsymbol{X}_1^s = \boldsymbol{x}\right] = \begin{cases} \left\{\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{Y}_1), \boldsymbol{X}_1^s = \boldsymbol{x}\right\}, & \boldsymbol{x} \le \boldsymbol{t}, \\ 0, & \boldsymbol{x} > \boldsymbol{t}. \end{cases} \tag{58}$$

We, again, only allow for two shot outcomes: either the shot hit or it missed. Thus, we describe the state of the process after the first shot by

$$\left\{\boldsymbol{P}(\boldsymbol{X}_1^h \le \boldsymbol{t}) \mid \boldsymbol{I}(\boldsymbol{Y}_1), \boldsymbol{X}_1^s = \boldsymbol{x}\right\} = \begin{cases} \boldsymbol{M}(\boldsymbol{t} - \boldsymbol{x}), & \boldsymbol{I}(\boldsymbol{Y}_1) = 1, \\ 1, & \boldsymbol{I}(\boldsymbol{Y}_1) = 0. \end{cases} \tag{59}$$

However, now process outcome also depends on the survivability response.

For each survivability response, we allow only two possible outcomes: success or failure. Again, if we turn to Doob (1994), we can define the probability of a successful response as

$$\boldsymbol{p}_w = \int_{\boldsymbol{I}(\boldsymbol{W}_i)=1} \boldsymbol{dF}_W, \tag{60}$$

where $\boldsymbol{F}_W$ is the distribution function for the employment of our survivability response. For purposes of this paper, we assume that $\boldsymbol{p}_w$ is both a fixed and known quantity. Similarly, we find the probabilities of hit conditioned on the success or failure of the survivability response as

$$\begin{aligned} \boldsymbol{p}_{h|s} &= \int_{\boldsymbol{I}(\boldsymbol{Y}_i)=1} \boldsymbol{dF}_{Y|\boldsymbol{I}(\boldsymbol{W}_i)=1}, \\ \boldsymbol{p}_{h|f} &= \int_{\boldsymbol{I}(\boldsymbol{Y}_i)=1} \boldsymbol{dF}_{Y|\boldsymbol{I}(\boldsymbol{W}_i)=0}. \end{aligned} \tag{61}$$

Again, we assume that both $\boldsymbol{p}_{h|s}$ and $\boldsymbol{p}_{h|f}$ are fixed and known quantities. We can combine eqs 60 and 61 to find the probability that the outcome of a given shot was a hit by applying the law of total probability

$$\boldsymbol{p}_{hw} = \boldsymbol{p}_{h|s}\boldsymbol{p}_w + \boldsymbol{p}_{h|f}(1 - \boldsymbol{p}_w) \tag{62}$$

where p_{hw} denotes the probability of hit in the presence of survivability measures. Now, eq 58 becomes, after applying the law of total probability over the range shot outcomes and combining the result with eq 59

$$\left[M(t) \mid X_1^s = x\right] = \begin{cases} p_{hw} + (1 - p_{hw})M(t-x), & x \le t, \\ 0, & x > t. \end{cases} \tag{63}$$

Then, by applying the law of total probability over the distribution of shot intervals, we obtain

$$\begin{aligned} M(t) &= \int_0^\infty \left[M(t) \mid X_1^s = x\right] dF_X^s(x) \\ &= \int_0^t \left[p_{hw} + (1 - p_{hw})M(t-x)\right] dF_X^s(x) \\ &= p_{hw}G_X^s(t) + (1 - p_{hw})\int_0^t M(t-x)\, dF_X^s(x). \end{aligned} \tag{64}$$

We will leave eq 64 as the principal result of this section; however, one asks, "what about detection?" We address that question via an example. Wand et al. (1993) introduced the process of detection into a stochastic tank duel. In their duel, they suggested that the negative exponential density adequately modeled the time to detect; thus, we can completely characterize the process of detection by one parameter, the mean time to detect. If we assume that the distribution $G_X^s(t)$ in eq 64 solely represents detection, and that it is a negative exponential distribution, then the survivability of our platform is a function of the mean time to detect. Hence, increasing survivability implies that we delay detection via camouflage or other means. Therefore, without any further development, we can leave eq 64 as our representation of the generalized marksman problem with survivability.

7.2 A Simple Damage Model with Survivability Measures

We now consider the second half of the combatant-target interaction, a process we modeled via our damage process. In eq 30, we developed a damage model that, in a rudimentary fashion, incorporated the effects of armor. In this model, the effect of the armor on an incoming projectile was all or nothing. That is, if the projectile perforated the armor, then the target was rendered mission incapable, but, if the projectile did not perforate the armor, then the target sustained no damage. We now wish to model survivability measures that augment armor protection by interfering with the projectile, for example an active protection system. Again, $\{S_k^h, I(A_k), I(R_k)\}$ denotes our stochastic process; however, we augment the notation of section 3.3 by introducing R to reflect that the process also depends on the survivability response. We employ the indicator function in the following sense

$$I(R_k) = \begin{cases} 1, & \text{the survivability measure interfered with the shot,} \\ 0, & \text{the survivability measure did not interfere with the shot.} \end{cases}$$

For this process, hit epochs occur at $\{S_k^h\}_{k=1}^{\infty}$, and the epochs of hits that render a target mission incapable occur at $\{S_l^d\}_{l=1}^{\infty}$. X_k^h and X_l^d will take the same meanings as section 3.3. We assume that the distribution of X_k^h is either a given, or is obtained from the generalized marksman process (eq 24 or 64 as appropriate). We assume that $I(A_k)$ and $I(R_k)$ are independent of the intervals, X_k^h, and set the goal for this section as finding X_l^d in terms of X_k^h and the survivability response. Because we assumed that the survivability response and the effect of the armor, to reach our goal we need only find an expression for p_k in eq 30.

First, we define some probabilities. Recall from section 3.3, that when a projectile perforated the target's armor, it caused lethal damage. This is still true; however, now the likelihood that a projectile perforates the armor is altered by the survivability response. In section 3.3, eq 30 modeled the effect of armor, and equated p_k to armor effectiveness. We now alter that equation by incorporating the survivability response

$$\begin{aligned} p_{A|s} &= \int_{I(A_k)=1} dF_{A|I(R_k)=1}, \\ p_{A|f} &= \int_{I(A_k)=1} dF_{A|I(R_k)=0}, \end{aligned} \tag{65}$$

where s and f denote, respectively, the success or failure of the survivability response. Given that a projectile will hit almost surely, we deploy a survivability response; a response that is either successful or not. Thus, we define

$$p_R = \int_{I(R_k)=1} dF_R. \tag{66}$$

to be the probability that the survivability response successfully interfered with the incoming projectile. In eq 66, F_R is the distribution of outcomes for the survivability response. By combining eqs 65 and 66, we can now compute p_k as

$$p_k = p_{A|s}\, p_R + p_{A|f}\,(1 - p_R). \tag{67}$$

Now, we refer to section 3.3 and immediately write the solution to the duel with armor and damage

$$M(t) = p_k F_X^h(t) + (1 - p_k)\int_0^t M(t-x)\, dF_X^h(x) \tag{68}$$

where we invoked the renewal argument $M(t) = P(X_d^h \le t)$. Consequently, eq 68 becomes the primary result of this section, and we note that $F_X^h(t)$ may either be a given or obtained from eq 24, or its analog with survivability eq 64.

8. Conclusions

In this paper, we studied a fundamental duel composed of two marksmen. Initially, we turned to renewal theory in an effort to reformulate the marksman problem of Williams and Ancker (1963). We then generalized the marksman problem to include the process of target detection developed by Wand et al. (1993). Finally, we developed a rudimentary damage model that included the effect of platform armor. Thus, having established the pieces of a renewal duel model, we suggested that one could view the fundamental duel as a series system with two components. With this idea, we turned to David and Moeschberger (1978) and created the fundamental duel via their method of competing risks. Initially, we adopted their method as an alternative means of computing the metric we desired: the probability of survival. While we achieved the immediate goal of defining the probability of survival in terms of each combatant, we required an assumption of independence between combatants that we hoped to avoid. We justified independence with the work of Tsiatis (1975), Peterson (1976) and Miller (1977); however, we also realized that need for this assumption was implied by the notion of a passive target. We now realize that the assumption of a passive target can be relaxed; however, the result is a competing risk model with dependent life times. While later work will address this issue, for now we highlight the work of Langberg et al. (1978). Langberg et al. studied the problem of converting a model with dependent lifetimes into an independent equivalent while still preserving essential qualities; hence, we concluded it may be possible to overcome the requirement for independence. Finally, we returned the concept of layered survivability to establish the basis for the remainder of the paper. There we related our generalized marksman problem and the damage process to layered survivability, and concluded by adding survivability to the renewal duel model.

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